DETERMINATION OF MODAL RESIDUES AND RESIDUAL FLEXIBILITY FOR TIME-DOMAIN SYSTEM REALIZATION

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ABSTRACT

A linear least-squares procedure for the determination of modal residues using time-domain system realization theory is presented. The present procedure is shown to be theoretically equivalent to residue determination in realization algorithms such as the Eigensystem Realization Algorithm (ERA) and Q-Markov COVER. However, isolating the optimal residue estimation problem from the general realization problem affords several advantages over standard realization algorithms for structural dynamics identification. Primary among these are the ability to identify data sets with large numbers of sensors using small numbers of reference point responses, and the inclusion of terms which accurately model the effects of residual flexibility. The accuracy and efficiency of the present realization theory-based procedure is demonstrated for both simulated and experimental data.

I. INTRODUCTION

Research in structural identification in recent years has lead to a proliferation of algorithms based upon system realization theory (Ho and Kalman, 1965; Juang, 1987). These system identification techniques, such as the Eigensystem Realization Algorithm (ERA) (Juang and Pappa, 1985), ERA with Data Correlations (ERA-DC) Juang, et al., 1988), and the Q-Markov COVER (QMC) (King, et al., 1988), have important direct applications to structural control, such as identification and order reduction of input-output models for robust control and adaptive on-line identification for nonlinear systems control. These algorithms all realize a model by minimizing some measure of the difference between the measured and reconstructed discrete-time impulse response functions, heretofore referred to as Markov parameters. In contrast to many classical modal identification techniques, the system realization algorithms are time domain techniques and are generally applicable to multiple-input multiple-output (MIMO) measured data systems.

These algorithms have arguably attracted the most attention for their use in modal test data analysis and reduction for identification of structural parameters (Pappa and Juang, 1988; Liu and Skelton, 1993). There are a number of reasons for this. First, these methods are fairly simple to understand and implement, requiring only standard matrix manipulation and numerical analysis functions such as those available in MATLAB. Secondly, these methods are founded on sampled data systems theory, which is directly applicable to inexpensive microprocessor-based data acquisition systems. Finally, system realization theory offered a simplification of the modal identification process by providing a clear indication (at least ideally) of dynamic order and by unifying the pole identification and residue estimation problems into a single step analysis. In other words, these methods were powerful tools at the right time and a practical approach for engineers unfamiliar with existing modal parameter identification methods and research.

These popular realization algorithms have, however, lacked the practical capabilities inherent in many standard modal identification software packages. Although these packages use some multiple reference time domain identification techniques, such as Polyreference (Vold, et al., 1982), they also feature separate treatment of pole identification and residue estimation, and the capability to estimate residual flexibility and inertia which improve model reconstructions. By contrast, ERA and other system realization theory-based techniques identify simultaneously the poles and residues in a unified model, and do not generally provide for the modeling of residual effects. Furthermore, many researchers have noted problems in achieving highly accurate reconstructions of some types of modal data using discrete time-domain realization algorithms, which has lead, among other things, to the development of frequency domain-based realization techniques (Jacques and Miller, 1993; Horta and Juang, 1993) and residue re-estimation (Mayes, 1993; Peterson and Alvin, 1994).

The purpose of this paper is to develop additional practical capabilities for modern time domain realization-based algorithms (such as ERA) through system realization terminology. As such, we intend the present paper to provide a natural complement to existing system realization literature. Our approach is based on a time domain estimation of the modal residues and residual flexibility, given a prior identification of the poles (i.e. frequencies and damping rates) and the modal participation factors of the system inputs. That is, for the discrete-time state space model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

our approach determines C and D given that A and B have been identified in a prior analysis, using perhaps a subset of the measured response functions. We will show how this estimation is consistent with, and related to, the residue estimation implicit in existing system realization algorithms. The procedure also provides a time-domain alternative to the approach of re-estimating of the mode shapes in the frequency domain using frequency response functions (Mayes, 1993; Peterson and Alvin, 1994).

To this end, the paper is organized as follows. In Section II, the time domain-based system realization theory and procedure is presented. In Section III, a procedure for optimally computing the mode shape matrix *C* using a linear least-squares solution with *A* and *B* from Eqn. 1 is detailed, and its relationship to the ERA computation of *C* is examined. In Section IV, the present mode shape estimation procedure is utilized to develop three useful generalizations of ERA for identification of structural dynamic models. Section V applies the procedure to a realistic simulated data example, and to experimental data. Conclusions are offered in Section VI.

II. REVIEW OF TIME DOMAIN SYSTEM REALIZATION PROCEDURE

We begin by presenting the governing equations of motion for structural dynamics in their usual forms. The response of a structure to a set of forces or inputs *u(t)* is usually modeled as a spatially discretized second order matrix differential equation of the form:

$$M\ddot{q} + D\dot{q} + Kq = \hat{B}u \tag{2}$$

where *M* is the mass matrix, *D* is the damping matrix, *K* is the stiffness matrix, and $\hat{B}$ is the force influence matrix. The vector *q(t)* includes the physical degrees of freedom (DOF) of the model. If we define the *n* associated normal modes Φ_n of Eqn. 2 according to:

$$K\Phi_n = M\Phi_n\Omega \tag{3}$$

$$\begin{aligned} \Phi_n^T K \Phi_n &= \Omega = \{\omega_{ni}^2, i = 1 \ldots n\} \\ \Phi_n^T M \Phi_n &= I_{n\times n} \\ \Phi_n^T K \Phi_n &= \Xi \end{aligned} \tag{4}$$

then the structural model can be placed into the first order modal state-space form:

$$\begin{aligned} \dot{x}_\eta &= A_\eta x_\eta + Bu \\ y &= C_\eta x_\eta + Du \end{aligned} \tag{5}$$

in which *y(t)* is the response, and the state-space matrices are:

$$A_\eta = \begin{bmatrix} 0 & I \\ -\Omega & -\Xi \end{bmatrix} \qquad B_\eta = \begin{bmatrix} 0 \\ \Phi_n^T\hat{B} \end{bmatrix}$$
$$C_\eta = \begin{bmatrix} H_d & 0 \end{bmatrix} + \begin{bmatrix} H_v & 0 \end{bmatrix} A_\eta + \begin{bmatrix} H_a & 0 \end{bmatrix} A_\eta^2 \tag{6}$$

in which H_d, H_v, and H_a are the output displacement, velocity and acceleration location influence arrays, respectively.

Because experimental vibration data is sampled in time, time domain linear system realization procedures begin from the presumption that a finite order discrete state-space model of the system exists of the form:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{7}$$

in which *k* is the time sample index. The procedure by which Eqn. 5 is sampled to lead to Eqn. 7 must be done carefully to avoid illconditioning due to the transformation from the continuous *(s)* plane to the discrete *(z)* plane. Likewise, the transformation from a realized model of the form of Eqn. 7 back to the continuous representation of Eqn. 5 requires careful eigenrotation and mass normalization, as described by Alvin and Park (1994).

When the model of Eqn. 7 is used as a predictor, the arbitrary response to an input *u(k)* is given by:

$$y(k) = \sum_{i=1}^{k} M(k-i)u(i) \qquad 1 \le k < \infty \tag{8}$$

in which the system Markov parameters *M(k)* are related to the state-space matrices by

$$M(k) = \begin{cases} D & k = 0 \\ CA^{k-1}B & k > 0 \end{cases} \tag{9}$$

All state-space time domain realization methods attempt to find the state space matrices *A, B, C, D* from measurements of the sequence *M(k)*. This is the process known as system realization.

The essential considerations in system realization are the selection of the model order (it is presumed that the model form is correct) and the determination of the state space parameters from a minimization of some prediction error. For ERA, the prediction error is defined in terms of a Hankel matrix of the Markov parameters, as defined by:

$$H_{rs}(k) = \begin{bmatrix} M(k+1) & M(k+2) & \ldots & M(k+s) \\ M(k+2) & M(k+3) & \ldots & M(k+s+1) \\ \vdots & \vdots & \ldots & \vdots \\ M(k+r) & M(k+r+1) & \ldots & M(k+r+s-1) \end{bmatrix} \tag{10}$$

The ERA realization finds the linear least squares solution to minimize the error in the shift in the Hankel matrix of the system model and the data according to:

$$H_{rs}(k-1) = V_r A^{k-1} W_s \tag{11}$$

in which

$$V_r = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix} \quad W_s = \begin{bmatrix} B \; AB \; \dots \; A^{s-1}B \end{bmatrix} \tag{12}$$

If the Hankel matrix is formed from the data, then the factors V_r and W_s are obtained from a singular value decomposition (SVD) of the 0-th Hankel matrix according to:

$$\begin{aligned} H_{rs}(0) &= \tilde{P}_r S_{rs} \tilde{Q}_s^T \\ V_r = \tilde{P}_r S_{rs}^{1/2} \qquad & W_s = S_{rs}^{1/2} \tilde{Q}_s^T \end{aligned} \tag{13}$$

The model order is selected (in principle) by examining the numerical rank of $H_{rs}(0)$. From this, the system realization problem is solved by:

$$\begin{aligned} A &= S_{rs}^{-1/2} \tilde{P}_r^T H_{rs}(1) \tilde{Q}_s S_{rs}^{-1/2} \\ B &= S_{rs}^{1/2} \tilde{Q}_s^T (1{:}n_x, 1{:}n_u) \\ C &= \tilde{P}_r (1{:}n_y, 1{:}n_x) S_{rs}^{1/2} \\ D &= M(0) \end{aligned} \tag{14}$$

Peterson (1992) discusses the computationally more efficient approach of factoring $H_{rs}(0)H_{rs}^T(0)$ instead of $H_{rs}(0)$ to obtain the factors of Eqn. 13. In this case, it is more computationally efficient to calculate the factors using a symmetric eigensolver in place of the SVD. By only computing the largest n_x eigenvalues and vectors of the Hankel matrix product, it is possible to determine realizations using very large values of *r* and *s* without calculating the entire spectral decomposition.

III. TIME DOMAIN RESIDUE ESTIMATION

Using the terminology consistent with system realization theory and outlined in Section II, we now develop the time domain residue estimation as follows.

Least-Squares Solution for Residues

Suppose the state space matrices A and B have been determined from a data set using, for example, ERA/DC or Q-Markov COVER. Then, using Eqn. 9, we have

$$\begin{aligned} CB &= M(1) \\ CAB &= M(2) \\ &\vdots \\ CA^{s-1}B &= M(s) \end{aligned} \tag{15}$$

Hence,

$$C\tilde{W}_s = \{M(1{:}s)\} \tag{16}$$

where

$$\begin{aligned} \tilde{W}_s &= \begin{bmatrix} B \; AB \; \dots \; A^{s-1}B \end{bmatrix} \\ \{M(1{:}s)\} &= \begin{bmatrix} M(1) \; M(2) \; \dots \; M(s) \end{bmatrix} \end{aligned} \tag{17}$$

The least squares solution for *C* using Eqn. 16 is given by

$$\begin{aligned} C &= \{M(1{:}s)\}\tilde{W}_s^+ \\ &= \{M(1{:}s)\}\tilde{W}_s^T[\tilde{W}_s\tilde{W}_s^T]^{-1} \end{aligned} \tag{18}$$

The solution is well-defined as long as $s > n$, where *n* is the system order (dimension of *A*). Note that the solution for *D*, that is

$$D = M(0) \tag{19}$$

holds under the present theory, so long as additional terms, such as residual flexibility, are not added to the problem.

The implementation of Eqn. 18 is straightforward because we can utilize the singular value decomposition of $H_{rs}(0)$ used in the previous analysis to realize *A* and *B* in order to determine $\tilde{W}_s^+$:

$$\tilde{W}_s^+ = Q_s S_{rs}^{-\frac{1}{2}} \tag{20}$$

Thus,

$$C = \{M(1{:}s)\} Q_s S_{rs}^{-\frac{1}{2}} \tag{21}$$

Relationship to Residue Estimation in ERA

As reviewed in Section II, in ERA the solution for *C* is given by

$$C = E_l \tilde{V}_r = E_l P_r S_{rs}^{\frac{1}{2}} \tag{22}$$

where $E_l = \begin{bmatrix} I_{l \times l} \; 0 \; \dots \; 0 \end{bmatrix}$ and $\tilde{V}_r$ is the generalized observability matrix, which is realized from the singular value decomposition of the measured Hankel matrix $H_{rs}(0)$, viz.

$$H_{rs}(0) \approx \tilde{H}_{rs}(0) = \tilde{V}_r \tilde{W}_s = P_r S_{rs} Q_s \tag{23}$$

Then, from Eqn. 22 we have

$$\begin{aligned} C &= E_l \tilde{V}_r \tilde{W}_s \tilde{W}_s^+ = E_l \tilde{H}_{rs}(0) \tilde{W}_s^+ \\ &= \{\tilde{M}(1{:}s)\}\tilde{W}_s^+ \end{aligned} \tag{24}$$

Thus, comparing Eqn. 18 and Eqn. 24, the least squares solution for *C* is fully consistent with system realization theory-based residue determination. The fundamental distinction is that the ERA solution for *C* is optimal for the “realized” Markov parameters; that is, the approximated Markov parameters as expressed by the realized Hankel matrix $\tilde{H}_{rs}(0)$, whereas the least squares solution is optimal for the actual measured Markov parameters.

IV. ALGORITHMS BASED ON PRESENT THEORY

An Eigensystem Realization Algorithm using Reference Point Responses (ERA-RP)

The residue estimation algorithm presented in Section III leads to a very useful generalization of ERA for structural dynamics identification. Since the modes which are identifiable from the data are limited to those which are disturbable from the system inputs, it is only necessary to include a small number of reference point responses from which the same modes are observable. In the case of structural dynamics when the system is reciprocal (i.e. symmetric mass, stiffness and damping properties), the logical sensor complements are driving point measurements, that is sensors co-located with the system input degrees of freedom.

The prime advantage of this approach is that it enables the use of longer data records for the same Hankel matrix dimension, or allows the reduction of the Hankel matrix dimension to increase overall computational efficiency. The use of longer data records is important for obtaining accurate and consistent frequency and damping estimates from real data. Reducing the size of the Hankel matrix is also important because the major computational overhead in the system realization procedure is strongly dictated by the minimum dimension of the matrix.

For example, suppose a typical modal test of a complex structure is performed for the purpose of characterizing the normal modes. If the test is measured using 100 accelerometers and 3 force inputs, an ERA analysis might utilize Hankel block dimensions of $r = 50$, $s = 2000$, leading to a Hankel matrix of size 5000×6000. On the other hand, a reference point ERA analysis might instead use $r = 500$, $s = 1600$, for a Hankel matrix of size 1500×4800. In the latter case, the length of the Markov sequence actually used in the Hankel matrix is slightly greater than in the ERA analysis, but the minimum matrix dimension is reduced by 70% and the computational effort required to decompose the matrix is reduced by approximately 97%.

Recomputing Residues after Elimination of Inaccurate Poles

The experimental study by Doebling, et al. (1994) found two main problems with determining structural poles from time domain realization algorithms:

- Many structural poles converge only after massive overspecification of the model order (n_x in Eqn. 14). Overspecification of model order, however, engenders additional computational or noise modes which should not be retained for subsequent analysis using the identified model.
- Poles which have converged can occasionally split into two nearly repeated (but nonphysical) modes as model order overspecification is increased to converge other less observable poles.

In view of these pathologies, it is usually necessary to use one or several quantitative model quality indicators (MQI) to detect convergence and discriminate unwanted or unreliable modes from the system realization (Peterson and Alvin, 1994).

Unfortunately, because all the global mode shapes are extracted through the realization process simultaneously, only the mode shapes of the full realization can be considered to be optimal in any sense. If, however, some modes are not retained for further analysis, the remaining mode shapes cannot be said to be optimal with respect to either the measured or the realized response parameters. This is often not a problem if the modes are well-spaced and orthogonal via the measurement points. It can be a serious problem, however, when computational mode splitting, as described above, occurs in the realization analysis. In this case, the mode shape information can split between the two nearly identical poles. Hence, when splitting is detected and one pole is removed from the modal set, important mode shape information is also lost.

The application of the linear least-squares solution for C is straightforward in this case. Simply perform the system realization analysis to obtain A, B and C in their decoupled modal form. Then, after computing various MQI and removing unreliable poles from A and B, C is recomputed using Eqn. 18. It should be noted that the generalized controllability matrix W_s must be recomputed using the reduced A and B matrices, rather than using the singular values and vectors of the Hankel matrix as in Eqn. 21. This is not a significant computational burden, however, as powers of A are inexpensive to compute in the decoupled block modal form, and the largest matrix inverse operation is of the order of the retained modes (which is relatively small in most instances).

Inclusion of Residual Flexibility Terms

One serious deficiency of the discrete-time state space model form common to ERA and other algorithms is that it cannot always account for the residual flexibility effects of modes outside the measurement bandwidth. In particular, when using velocity sensors or accelerometers (arguably the most popular transducer types for modal testing), the modes above the measurement bandwidth contribute a sum term proportional to the Laplace terms s and s^2, respectively (Ewins, 1984; Peterson and Alvin, 1994). Such terms cannot be properly expressed in the discrete-time state-space model form, however, even though their influence is captured in the measured FRFs (and thus the Markov parameters).

One possible corrective approach is to compute a residual flexibility term by fitting the trend of the frequency domain error between the measured FRF and its model-based reconstruction. This approach is generally effective but ignores the weak coupling at all frequencies between the contribution of the identified modes and residues and the residual flexibility. The result also mixes least-squares time-domain and frequency domain computations, obscuring the optimality criterion of the complete model response.

The present linear least-squares algorithm for estimating mode shapes of the system realization is easily extended to include the residual flexibility contribution in a consistent manner. Starting from the proper expression of the discrete FRF including residual terms, we have

$$G(f_k) = C\left(e^{\frac{j2\pi k}{N}} I - A\right)^{-1} B + D + \left(\frac{j2\pi k}{N\Delta t}\right)^{p} F \qquad (25)$$

where F is the residual flexibility matrix and p is the differentiation order of the sensor type with respect to displacement (i.e. p=1 for velocity, p=2 for acceleration). Here the Laplace term s has been evaluated along the frequency axis $j\omega$ at the discrete frequency values $\omega_k = 2\pi f_k$, where $f_k = k/(N\Delta t)$ and k, N, and Δt

are the sample index, total number of samples and sampling rate, respectively.

Taking the inverse discrete fourier transform (IDFT) of Eqn. 25 lead to the following relationships

$$\begin{aligned} M(0) &= D + M_{s^p}(0)F \\ M(1) &= CB + M_{s^p}(1)F \\ M(2) &= CAB + M_{s^p}(2)F \\ &\vdots \\ M(k) &= CA^{k-1}B + M_{s^p}(k)F \end{aligned} \tag{26}$$

where $M_{s^p}(i)$ are the IDFT coefficients of the discrete function

$$s_k^p = \left(\frac{j2\pi k}{N\Delta t}\right)^p \tag{27}$$

evaluated at

$$k = \left[1-\frac{N}{2}\ \dots\ -1\ 0\ 1\ \dots\ \frac{N}{2}-1\ \frac{N}{2}\right] \tag{28}$$

The order of values for k given in Eqn. 28 depends on the numerical algorithm for computing the IDFT; the above ordering is consistent with that used for the inverse fast Fourier transform in MATLAB. The time-domain coefficients $M_{s^p}(i)$ of the residual term are essentially normalized Markov parameters of the sum contribution of the modes above the measurement bandwidth to the estimated FRFs. Figure 1 shows the coefficients of s and s^2 for a small sample record with unit sample time. Using this result, we can then form the linear least-squares problem

$$\begin{aligned} \{M(1{:}s)\} &= \begin{bmatrix} C & F \end{bmatrix} \begin{bmatrix} \tilde{W}_s \\ \{M_{s^P}(1{:}s)I_m\} \end{bmatrix} \\ &= \begin{bmatrix} C & F \end{bmatrix} \hat{W} \\ M(0) &= D + M_{s^P}(0)F \end{aligned} \tag{29}$$

where I_m is the identity matrix, m are the number of inputs, and

$$\{M_{s^P}(1{:}s)I_m\} = \begin{bmatrix} M_{s^P}(1)I_m & M_{s^P}(2)I_m & \dots & M_{s^P}(s)I_m \end{bmatrix} \tag{30}$$

The solution of Eqn. 29 is then

$$\begin{aligned} \begin{bmatrix} C & F \end{bmatrix} &= M\{1{:}s\}\hat{W}^T(\hat{W}\hat{W}^T)^{-1} \\ D &= M(0) - M_{s^P}(0)F \end{aligned} \tag{31}$$

Note that when F and its coefficients are dropped from Eqn. 29, the solutions for C and D are given by Eqn. 16 and Eqn. 19, respectively.

The above formulation can often lead to an illconditioned matrix due to the mixture of continuous and discrete frequency domain terms. In order to avoid these problems, we replace s_k in Eqn. 27 and F in Eqn. 29 by frequency-normalized counterparts

$$\bar{s}_k = s_k\Delta t = \frac{j2\pi k}{N} \qquad \bar{F} = \frac{F}{(\Delta t)^p} \tag{32}$$

Then the estimated term $\bar{F}$ is multiplied by $(\Delta t)^p$ to obtain the correct residual flexibility term consistent with the continuous equations of motion.

V. APPLICATIONS AND EXAMPLES

Numerical Example using Modal Test Simulator

In order to properly understand the behavior of any system identification algorithm, it is important to perform simulations using data which is highly characteristic of the actual data the algorithm will ultimately be applied to. Often, in time domain system identification research, this realism is limited to the addition of gaussian noise to Markov parameters of displacement outputs, which are generated in the time domain by the discretized system equations. Unfortunately, this approach neglects the process by which Markov parameters are usually obtained in testing; that is, frequency domain FRF estimation of acceleration data with signal conditioning, ensemble averaging and digital signal processing. The signal processing and residual flexibility effects engendered can be far more significant on the performance of system realization algorithms than the level of noise which is typically encountered, at least in controlled modal testing environments. Therefore, a modal testing simulator was developed which includes all of the aforementioned effects, in addition to assumed measurement noise and burst random excitation.

Figure 2 shows a planar truss example. The model includes 36 unconstrained DOF, 18 acceleration sensors and 3 externally applied force inputs; 3 of the sensors are collocated with the 3 inputs. The modal testing simulator was used to generate the FRFs for all 54 input-output pairings. The simulator used 8192 samples per ensemble, sampled at 1000 Hz with anti-alias filtering set at 400 Hz. The FRFs were obtained using 10 ensemble averages and 1% noise was added to the measurements of the forces and accelerations.

The first stage of the time-domain system realization process consists of the estimation of the reliable system poles. For this example, the 3 driving-point (i.e. collocated output) measurements were retained as reference responses for a total of 9 FRFs. An efficient form of ERA (Peterson, 1992) was used with Hankel block dimensions of r=300 and s=2000 for total data matrix dimension of 900×6000 . If all response measurements had been included in the data matrix, the dimension would have been 5400×6000 . The order of the ERA-estimated model was varied from 50 to 100 states and the convergence of various MQI were studied. The final model order chosen was $n_x = 52$, for a nominal set of 26 modes. Of these, 3 modes (6 complex poles) were judged as inaccurate or unreliable and thus were removed from the modal set. A comparison of the retained modes to those of the exact model are shown in Table 1.

The second stage of the time-domain system identification consisted of using the present mode [illegible]e estimation algorithm to obtain mode shapes for the full set of 18 measured accelerometers. For the least-squares estimation, the first 2000 Markov parameters were used, consistent with the column dimension of the Hankel matrix used for the pole estimation. Figures 3 and 4 show the FRF reconstructions using the retained set of ERA modes together with the full mode shapes estimated using Eqn. 18. Before proceeding, we make the following observations.

First, the transfer FRF shown in Figure 4 cannot be obtained from the ERA analysis alone, as the Markov parameters for this input-output pair were not included in the Hankel matrix. That is, although the ERA-derived mode shapes did not include this response location, they were effectively estimated using the present procedure. Second, 3 modes were eliminated from the ERA realization, it was also useful to reestimate the mode shapes for the reference point responses, as the ERA mode shapes were extracted simultaneously with the inaccurate modes. In this particular case, because the inaccurate poles were not closely coupled with any of the retained poles, the original ERA mode shapes and the re-estimated mode shapes at the reference points were essentially identical.

Finally, other than the resonance which was not identified at approximately 312 Hz, the reconstructed FRFs are highly accurate at the resonance peaks. The zeros of the FRFs, however, show varying degrees of error, particularly the driving point response. These errors are due to the exclusion of a residual flexibility term in the ERA model and in the new mode shape estimation via Eqn. 17. If, however, we include a term to model residual flexibility, as in Eqn. 31, the reconstruction is significantly improved, as shown in Figure 5. Further improvement at low frequency could possibly be obtained by changing the number of time points used, or by applying constraints to Eqn. 25. The accuracy of the estimated mode shapes from Eqn. 18 and Eqn. 31 with respect to the exact mode shapes of the example model is shown in the last two columns of Table 1, in terms of the modal assurance criteria (normalized vector correlation). While the mode shape estimate without including the residual term is nearly exact, there is an improvement in the mode shape by simulataneously estimating the residual flexibility. This result is consistent with the existence of a weak coupling between the two response contributions.

<u>Application to Experimental Data</u>

Although the preceding numerical example was realistic, it is often helpful to verify the accuracy accrued by the curve fitting procedure through its application to actual experimentally measured data. Figure 6 shows a photograph of the three-dimensional cantilevered truss structure tested. The modal testing used one force input and 61 accelerometers (including a driving point locations), with a sampling frequency of 500 Hz and 50 ensemble averages.

As in the numerical example, the poles were estimated using ERA with the single driving point measurement; after selecting n_x =100 (50 modes), 28 modes were retained for the final model. The 61 response measurements were then re-estimated to yield the desired mode shapes and residual flexibility. A representative driving point and transfer FRF are shown in Figures 7 and 8. One important lesson learned with this data was that it was important to include the last 20 time samples of the impulse response in the least squares equation in order to obtain good estimates of the residual flexibility. This is because of the magnitude increase of M_{s^p} as $k \to N$ in Figure 1.

VI. CONCLUSIONS

A linear least-squares mode shape estimation algorithm using time domain system realization theory has been presented. The present procedure enhances existing time domain system realization algorithms such as ERA, ERA/DC and Q-Markov COVER by adding the ability to compute (or reestimate) global mode shapes when performing reference point-based pole estimation and unreliable pole elimination. Furthermore, the procedure can be generalized to estimate residual flexibility terms which cannot be modeled within the discrete state space form. These capabilities have been demonstrated via numerical and experimental data.

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TABLE 1: ACCURACY OF IDENTIFIED MODES USING RP-ERA WITH RESIDUAL FLEXIBILITY

Mode #	f_{exact} (Hz)	f_{ident} (Hz)	Error%	MAC
1	23.062	23.065	0.0113	1.0000
2	54.700	54.700	0.0011	1.0000
3	81.566	81.566	0.0004	1.0000
4	92.457	92.457	0.0003	0.9999
5	132.26	132.26	0.0005	1.0000
6	162.94	162.94	0.0028	0.9996
7	171.20	171.20	0.0002	1.0000
8	205.43	205.43	0.0002	1.0000
9	235.15	235.15	0.0006	1.0000
10	237.93	237.93	0.0003	1.0000

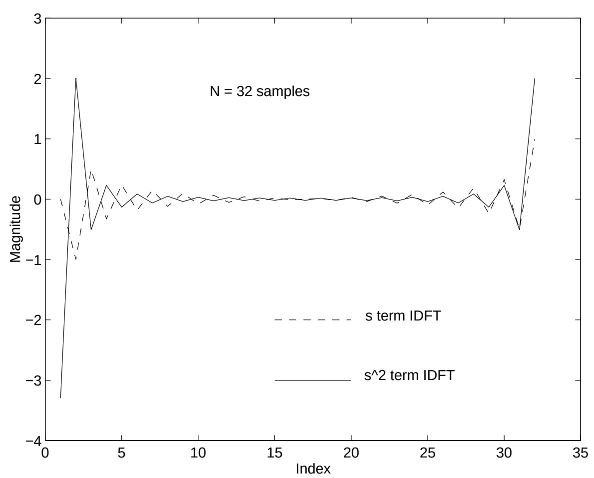

FIGURE 1: TIME DOMAIN REPRESENTATION OF THE LAPLACE TERMS *S* AND *S*^{2}

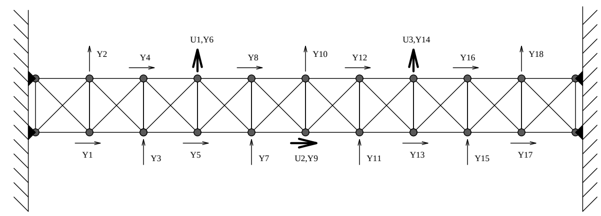

FIGURE 2: 2-D TRUSS NUMERICAL EXAMPLE

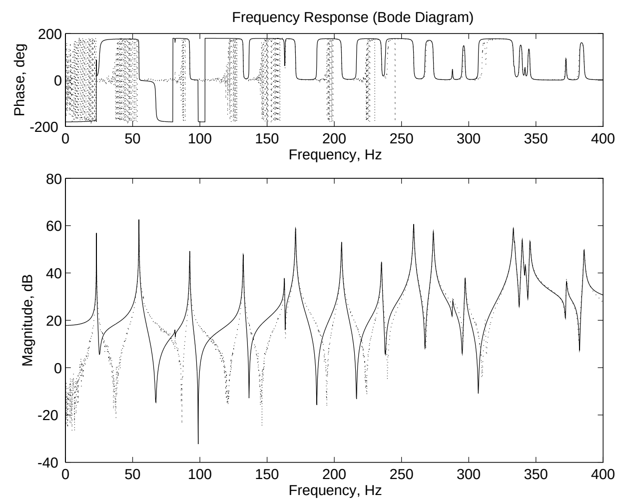

FIGURE 3: DRIVING POINT FRF RECONSTRUCTION

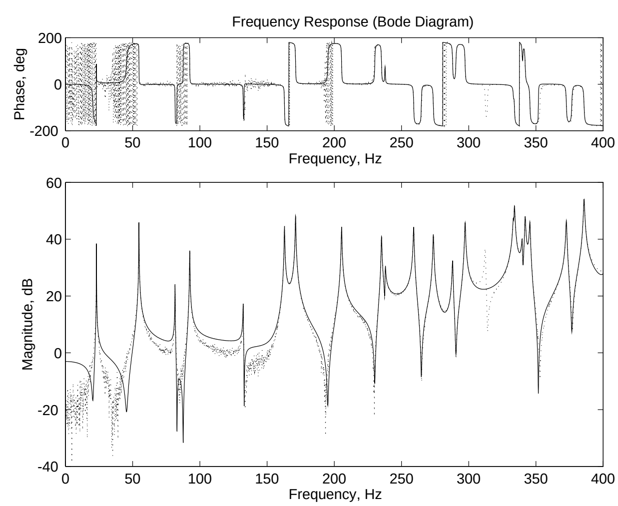

FIGURE 4: TRANSFER FRF RECONSTRUCTION

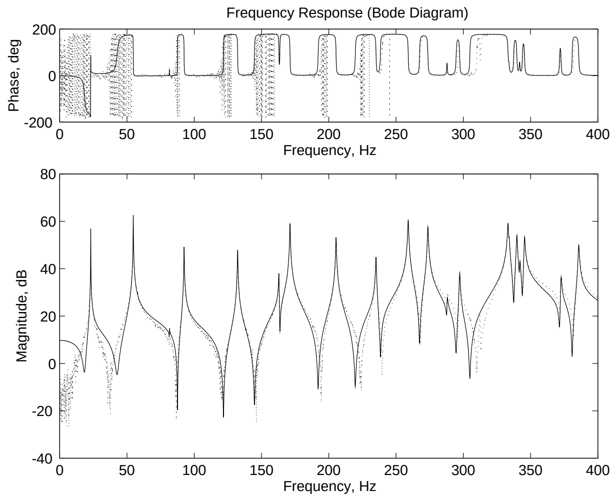

FIGURE 5: DRIVING POINT FRF WITH RESIDUAL FLEXIBILITY

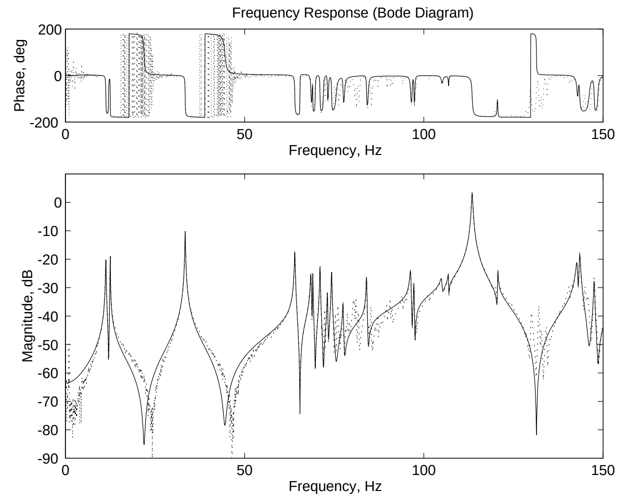

FIGURE 7: DRIVING POINT FRF AND RECONSTRUCTION FOR TRUSS TOWER

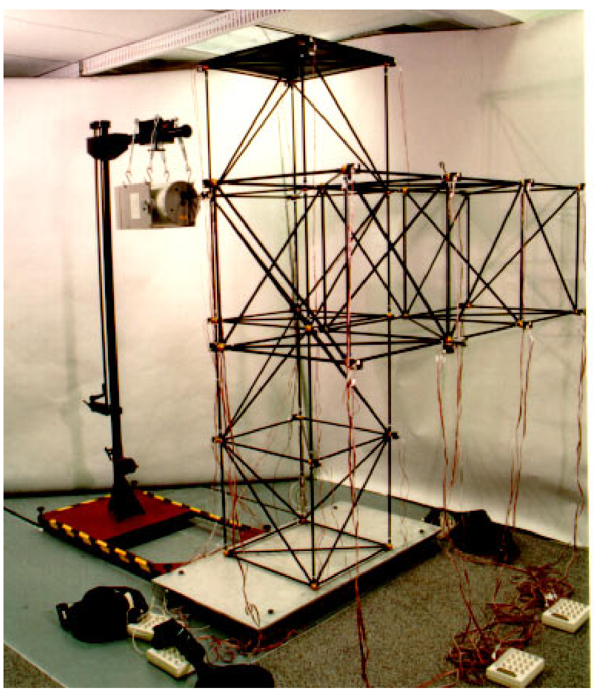

FIGURE 6: PHOTOGRAPH OF TRUSS MODAL TESTING

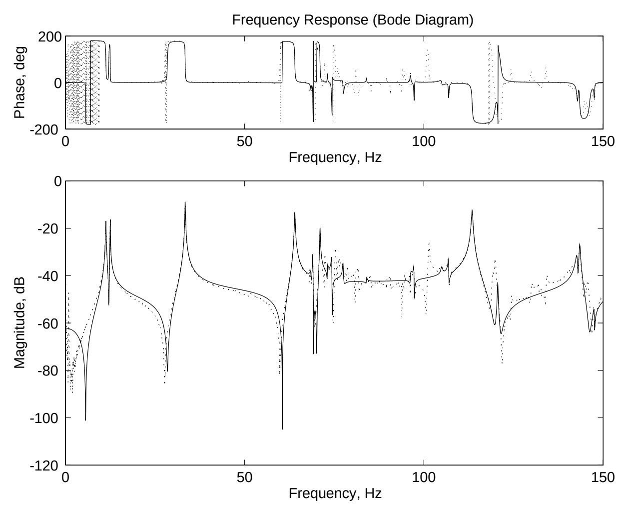

FIGURE 8: TRANSFER FRF AND RECONSTRUCTION FOR TRUSS TOWER